THE HYPERBOLA

The hyperbola is the basis for a navigation system called LORAN, (Long Range Navigation.) The hyperbola is the shape of the orbits of certain comets, and is also used to design telescopes and the cooling towers at nuclear power plants.

❑ THE MAIN EXAMPLE

Graph the hyperbola $x^2 - y^2 = 9$.

First, let's find the **domain**. We solve for y, and then analyze all the legal values of x:

$$x^2 - y^2 = 9 \Rightarrow -y^2 = -x^2 + 9 \Rightarrow y^2 = x^2 - 9 \Rightarrow y = \pm\sqrt{x^2 - 9}$$

Because of the square root, y is a real number only when the radicand, $x^2 - 9$, is greater than or equal to zero: $x^2 - 9 \geq 0$. You can solve this inequality by the Boundary Point Method. When you do, you will conclude that the domain of the graph is **$(-\infty, -3] \cup [3, \infty)$**. By the way, can you explain why the hyperbola is <u>not</u> a function?

Second, we'll check out **intercepts**. Since 0 is not in the domain, there cannot possibly be any y-intercepts. [Indeed, letting $x = 0$ gives $0^2 - y^2 = 9 \Rightarrow y^2 = -9 \Rightarrow y = \pm\sqrt{-9}$, not members of $\mathbb{R}$.] Thus, there are no y-intercepts. As for x-intercepts, let $y = 0$; this gives $x^2 - 0^2 = 9 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$. Therefore, there are two x-intercepts: **(3, 0)** and **(–3, 0)**.

Third, let's tackle **symmetry**. Replace x with $-x$: $(-x)^2 - y^2 = 9$, which is equivalent to $x^2 - y^2 = 9$, the original equation. Thus, the graph has y-axis symmetry. Similarly, you can show that the graph has x-axis symmetry. In fact, the graph of the hyperbola also has origin symmetry.

Fourth, we analyze **asymptotes**. Since there are no fractions in the hyperbola formula, it appears that the graph has no vertical asymptotes. To find any other possible asymptotes, we'll let $x \to \infty$ and see if y approaches anything interesting.

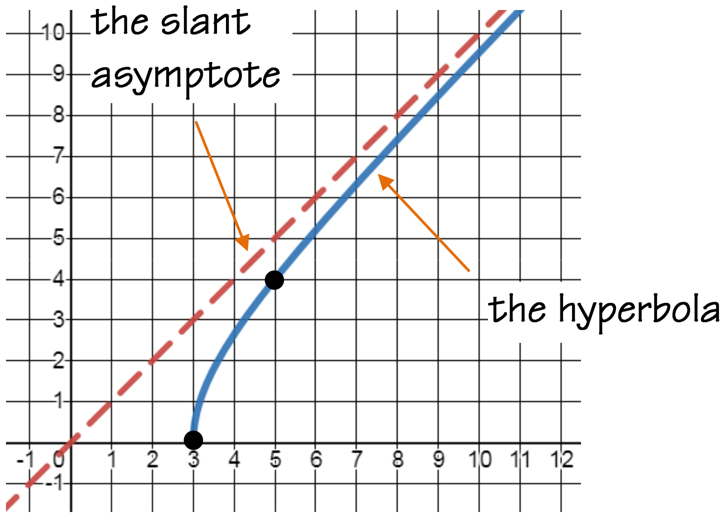
Let's construct a little x-y chart, and then choose really big values for x. Use your calculator to confirm these results.

x	y
5	± 4
10	± 9.5394
100	± 99.5499
1,000	± 999.9955
10,000	± 9,999.9996

What's going on here? Let's look just at the positive y-values. On the surface, it certainly appears that as x grows larger and larger, the y-values grow larger and larger. But look more carefully: The y-value is actually approaching the x-value. That is, as $\boldsymbol{x \to \infty, y \to x}$. We can write this limit as

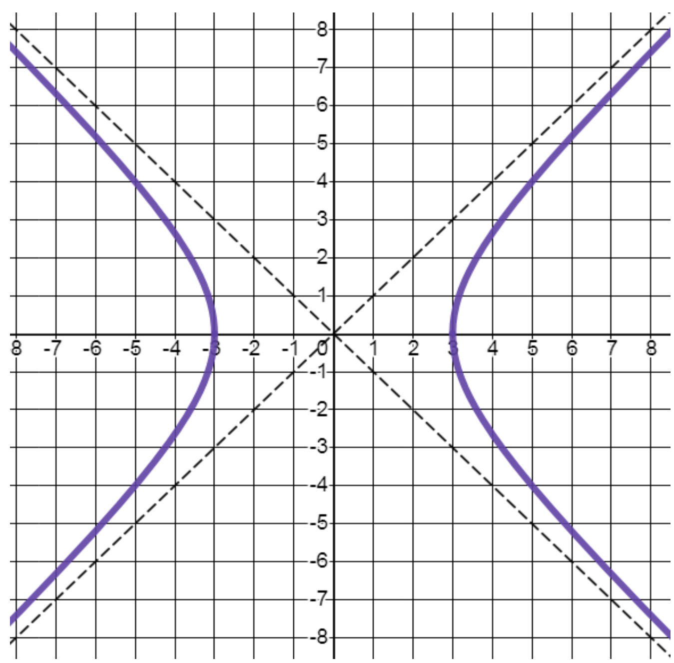
$$\lim_{x \to \infty} \sqrt{x^2 - 9} = x$$

This means that as x grows larger and larger, the curve is approaching the line $y = x$ (a 45° line passing through the origin). Since this line is neither vertical nor horizontal, we need a new adjective. We call this kind of asymptote a ***slant***, or ***oblique***, asymptote.



The analysis just completed gives a picture of the hyperbola in the first quadrant. It has an x-intercept at (3, 0), and a slant asymptote of $y = x$. Also, from the chart above, the hyperbola passes through the point (5, 4).

Since the hyperbola has y-axis symmetry, another piece of the hyperbola lies in the second quadrant. But the pieces of the hyperbola in Quadrants I and II have symmetric pieces in Quadrants III and IV, due to the x-axis symmetry of the hyperbola.



We're now ready to graph the entire hyperbola. Note that the dashed lines $y = x$ and $y = -x$ are the slant asymptotes of the hyperbola, and are graphed just for reference — they are not part of the hyperbola. Finally, the graph should convince you of the fact deduced earlier that the domain is $(-\infty, -3] \cup [3, \infty)$. In addition, the graph shows us that the **range** of the hyperbola is $\mathbb{R}$.

Homework

1. Regarding the main example above, $x^2 - y^2 = 9$, explain, using the formula, why it isn't a line, a parabola, a circle, or an ellipse.

2. Find the slant asymptotes for each hyperbola:

   a. $x^2 - y^2 = 25$ b. $x^2 - 9y^2 = 9$

   c. $36x^2 - y^2 = 36$ d. $9x^2 - 4y^2 = 36$

3. Find the domain and range of each hyperbola in the previous problem.

4. Perform a complete analysis (including a graph) on the hyperbola $x^2 - y^2 = 49$.

❑ *ANOTHER EXAMPLE*

Graph the hyperbola $y^2 - 4x^2 = 25$.

First we solve for y — this will allow us to find the domain and the slant asymptotes.

$$y^2 - 4x^2 = 25 \Rightarrow y^2 = 25 + 4x^2 \Rightarrow y = \pm\sqrt{25+4x^2}$$

Look at the radicand. No matter what kind of number x is, its square, x^2, is non-negative, which implies that $4x^2$ is non-negative, which implies that $25 + 4x^2$ is <u>at least</u> 25. So surely the radicand can <u>never</u> be negative. Thus, the **domain** of the hyperbola is $\mathbb{R}$.

Now it's time for the **intercepts**. Let $x = 0$; then $y^2 = 25$, or $y = \pm 5$. So we have a couple of y-intercepts: **(0, 5) and (0, –5)**. Now let $y = 0$; then $-4x^2 = 25 \Rightarrow x^2 = -\frac{25}{4} \Rightarrow$ no real solution for x. Therefore, there are no x-intercepts.

I'll leave it to you to verify that the hyperbola possesses all three types of **symmetry**.

To find the slant **asymptotes**, we refer to the form of the equation calculated above,

$$y = \pm\sqrt{25 + 4x^2}$$

Use your calculator to confirm the following ordered pairs (using just the positive square root.):

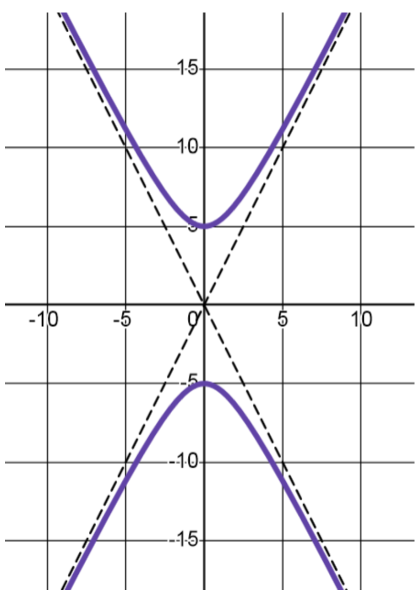
(10, 20.616) (100, 200.062) (1000, 2000.006)

As x grows larger and larger, the y-values appear to be growing closer and closer to whatever <u>twice x</u> is. That is, as $\boldsymbol{x \to \infty, y \to 2x}$. We write

$$\lim_{x \to \infty} \sqrt{25 + 4x^2} = 2x$$

Thus, one of the slant asymptotes is the line $\boldsymbol{y = 2x}$.

The graph goes through the intercept (0, 5) and then approaches the slant asymptote $y = 2x$. The x-axis and y-axis symmetries then give us the rest of the hyperbola in the other three quadrants. As before, the asymptotes (dashed) are graphed for reference.



Last, we discuss the **range** of the hyperbola. The y-values appear to be numbers that are either less than or equal to –5, OR greater than or equal to 5. That is, the range of the hyperbola is $\{y \in \mathbb{R} \mid y \le -5 \text{ or } y \ge 5\}$, which is $\boldsymbol{(-\infty, -5] \cup [5, \infty)}$.

Apollonius (Greek: 262 BC – 190 BC) gave the hyperbola its present name. See how the graph of the hyperbola seems to "fly away" in all directions? Look up the word "hyperbol<u>e</u>" in a dictionary (a big book that predates Google) and see if its meaning has any connection to the graph of a hyperbola.

Homework

5. Find the slant asymptotes for each hyperbola:

a. $y^2 - x^2 = 144$ b. $y^2 - 16x^2 = 16$

c. $9y^2 - x^2 = 9$ d. $25y^2 - 9x^2 = 225$

6. Find the domain and range of each hyperbola in the previous problem.

Perform a complete analysis (including a graph) on each hyperbola:

7. $y^2 - x^2 = 25$ 8. $y^2 - 9x^2 = 16$

9. $16y^2 - 9x^2 = 144$ 10. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

11. Classify each of the following as a line, parabola, circle, ellipse, or hyperbola:

a. $7x^2 - \pi x + 2y + 9 = 0$ b. $83x + 99y = 34$

c. $7x^2 + 2y^2 - x = 10$ d. $y^2 + 7y - x^2 - 2x - 100 = 0$

e. $7x + 99 = 400$ f. $32 - 9y + \pi = 0$

g. $\sqrt{2}\,x^2 + \sqrt{2}\,y^2 - \sqrt{e}\,x + \sqrt[3]{\pi}\,y - 10^6 = 0$

Review Problems

12. Classify as parabola, circle, ellipse, hyperbola, or none of them:

a. $x^2 + y^2 = -1$ b. $y^2 - x^2 = 5$ c. $2x^2 + 4y^2 = 7$

d. $x^2 - y = 3$ e. $2x^2 - \pi y^2 = \sqrt{2}$ f. $3y^2 = x^2 + 5$

13. a. Find the intercepts of $3x^2 - 2y^2 = 2$.
    b. Find the intercepts of $10y^2 - 30x^2 = 3$.
    c. Find the domain of $x^2 - 14y^2 = 144$.
    d. Find the range of $9x^2 - 36y^2 = 41$.
    e. Find the asymptotes of $49x^2 - y^2 = 10$.
    f. Find the asymptotes of $36y^2 - 4x^2 = 1$.
    g. Find all symmetries of $4x^2 - 9y^2 = 36$.

14. Find the intercepts, domain, range, asymptotes, and symmetry of the hyperbola $x^2 - y^2 = 4$. Graph the hyperbola.

15. Find the asymptotes and the range of $y^2 - 9x^2 = 16$. Sketch the graph.

16. True/False:

    a. Some of the hyperbolas in this chapter are functions.
    b. The domain of the hyperbola is $x^2 - y^2 = 9$ is $[-3, 3]$.
    c. Hyperbolas in this chapter always possess all three symmetries.
    d. The slant asymptotes for $16x^2 - 9y^2 = 1$ are $y = \pm\frac{4}{3}x$.
    e. There are no slant asymptotes for $25x^2 + 49y^2 = 1$.

f. The range of a hyperbola may consist of the union of two intervals.

g. Some of the hyperbolas in this chapter pass through the origin.

❑ *To ∞ and Beyond*

Find the **center** of the hyperbola

$$25x^2 - 9y^2 + 50x + 90y - 425 = 0$$

Hint: We note that the *origin* is the center of each hyperbola in this chapter.

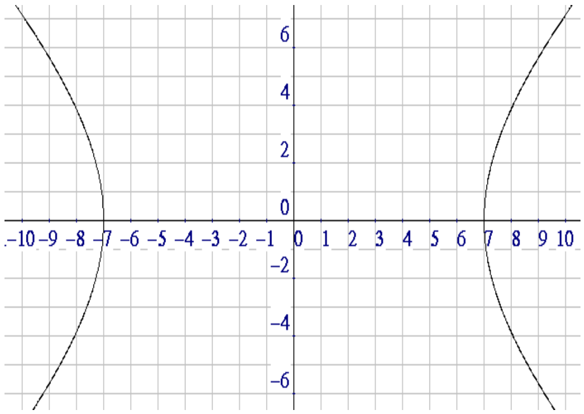
Solutions

1. Any letter squared $\Rightarrow$ it's not a line
Both letters squared $\Rightarrow$ not a parabola
Different coefficients on the squares $\Rightarrow$ not a circle
Opposite signs on the coefficients on the squares $\Rightarrow$ not an ellipse

2. a. $y = \pm x$ b. $y = \pm\frac{1}{3}x$ c. $y = \pm 6x$ d. $y = \pm\frac{3}{2}x$

3. a. $(-\infty, -5] \cup [5, \infty)$; $\mathbb{R}$ b. $(-\infty, -3] \cup [3, \infty)$; $\mathbb{R}$
c. $(-\infty, -1] \cup [1, \infty)$; $\mathbb{R}$ d. $(-\infty, -2] \cup [2, \infty)$; $\mathbb{R}$

4.



Domain = $(-\infty, -7] \cup [7, \infty)$

Range = $\mathbb{R}$ Not a function

No y-intercepts

x-intercepts: $(-7, 0)$ and $(7, 0)$

Symmetry: all three

$\lim_{x \to \infty} \sqrt{x^2 - 49} = x$

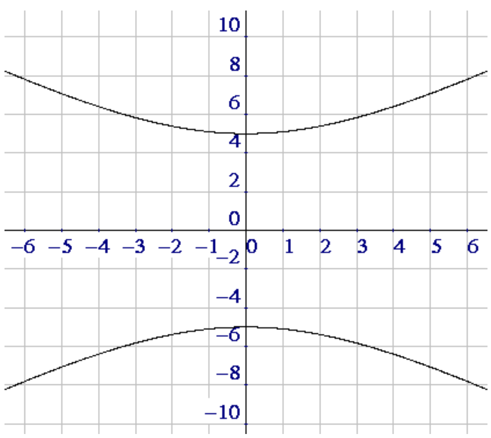
Slant asymptotes: $y = x$ and $y = -x$

5. a. $y = \pm x$ b. $y = \pm 4x$ c. $y = \pm \frac{1}{3}x$ d. $y = \pm \frac{3}{5}x$

6. a. $\mathbb{R}$; $(-\infty, -12] \cup [12, \infty)$ b. $\mathbb{R}$; $(-\infty, -4] \cup [4, \infty)$

c. $\mathbb{R}$; $(-\infty, -1] \cup [1, \infty)$ d. $\mathbb{R}$; $(-\infty, -3] \cup [3, \infty)$

7.

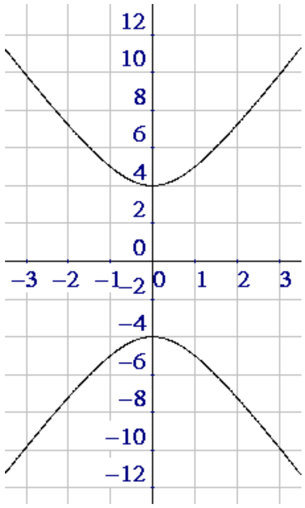


Domain = $\mathbb{R}$ Range = $(-\infty\ -5] \cup [5, \infty)$ Not a function

No x-intercept; y-intercepts: $(0, 5)$ and $(0, -5)$

Symmetry: all three Asymptotes: $y = x$ and $y = -x$

8.



Domain = $\mathbb{R}$

Range = $(-\infty, -4] \cup [4, \infty)$

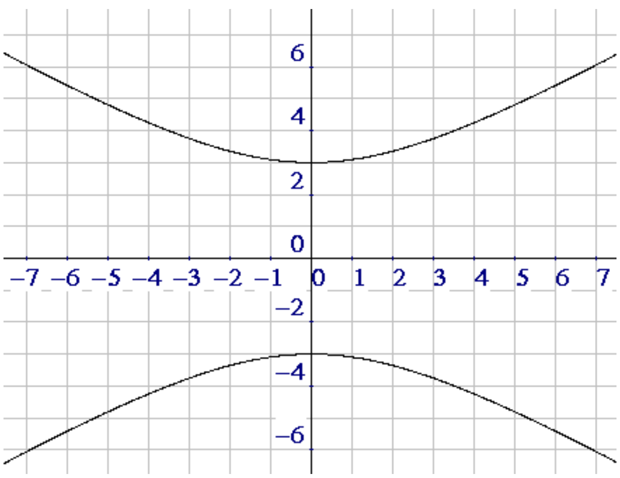
Not a function

No x-intercepts; y-intercepts: $(0, 4)$ and $(0, -4)$

Symmetry: all three

Asymptotes: $y = 3x$ and $y = -3x$

9.



Domain = $\mathbb{R}$

Range = $(-\infty\ -3] \cup [3, \infty)$

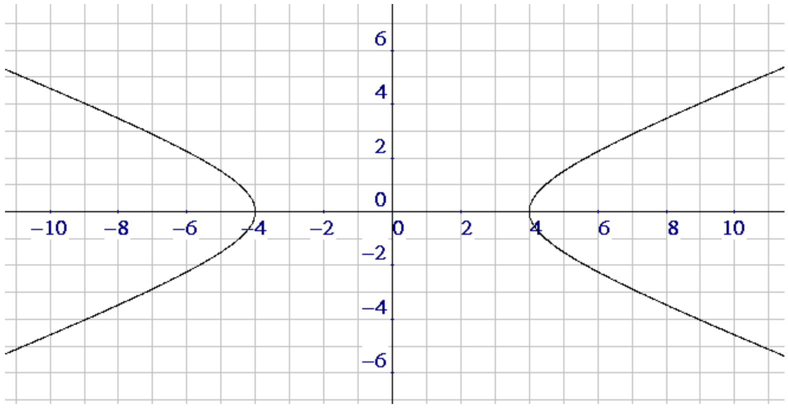
Not a function

No x-intercepts

y-intercepts: $(0, 3)$ and $(0, -3)$

Symmetry: all three

Asymptotes: $y = \pm\frac{3}{4}x$

10. 

Domain = $(-\infty, -4] \cup [4, \infty)$ Range = $\mathbb{R}$ Not a function

Intercepts: $(-4, 0)$ and $(4, 0)$ Symmetry: all three

Slant asymptotes: $y = \pm\frac{1}{2}x$

11. a. parabola b. line c. ellipse d. hyperbola

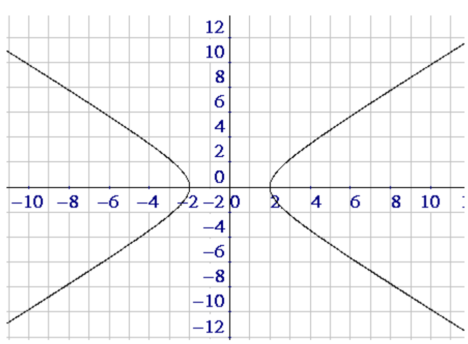
e. (vertical) line f. (horizontal) line g. circle

12. a. none b. hyperbola c. ellipse
d. parabola e. hyperbola f. hyperbola

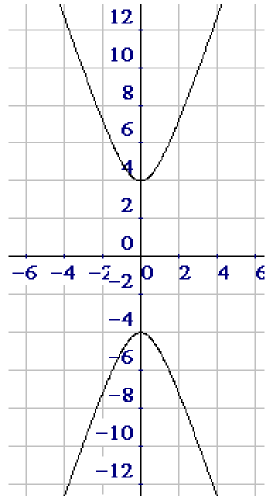
13. a. $\left(\pm\sqrt{\frac{2}{3}},\ 0\right)$ b. $\left(0,\ \pm\sqrt{\frac{3}{10}}\right)$ c. $(-\infty, -12] \cup [12, \infty)$ d. $\mathbb{R}$

e. $y = \pm 7x$ f. $y = \pm\frac{1}{3}x$ g. x-axis, y-axis, and origin

14. $(\pm 2, 0)$ Dom = $(-\infty, -2] \cup [2, \infty)$ Rg = $\mathbb{R}$ asym: $y = \pm x$
all 3 symmetries



15. $y = \pm\, 3x$; $(-\infty, -4] \cup [4, \infty)$



16. a. F b. F c. T d. T e. T f. T g. F

"Effort only fully releases its reward after a person refuses to quit."

Napoleon Hill